

MINIMAL HARMONIC VECTORS AND ERGODIC CONFORMAL MEASURES

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ABSTRACT. We establish a bijective correspondence between minimal harmonic vectors of a non-negative matrix and a class of conformal Borel measures that are ergodic for the shift on the path space defined by the matrix.

1. INTRODUCTION

Let Γ be a directed graph with a countable set Γ_E of edges (or arrows) and a countable set Γ_V of vertexes. We let $r(a) \in \Gamma_V$ be the range (or terminal) vertex of the edge a , and $s(a) \in \Gamma_V$ the source (or starting) vertex of a . Let $F : \Gamma_E \rightarrow \mathbb{R}$ be a function. For any $\beta \in \mathbb{R}$ we can then define a matrix $A(\beta) : \Gamma_V \times \Gamma_V \rightarrow [0, \infty]$ over Γ_V such that

$$A(\beta)_{v,w} = \sum_{a \in s^{-1}(v) \cap r^{-1}(w)} e^{-\beta F(a)}. \quad (1.1)$$

A vector (or function) $\psi : \Gamma_V \rightarrow [0, \infty[$ is $A(\beta)$ -harmonic when

$$\sum_{w \in \Gamma_V} A(\beta)_{v,w} \psi_w = \psi_v \quad \forall v \in \Gamma_V. \quad (1.2)$$

We are motivated by the following questions:

- A) For which β are there any non-zero $A(\beta)$ -harmonic vectors ?
- B) When there are non-zero $A(\beta)$ -harmonic vectors, what can we say about their structure, both individually and as a set ?

These questions come up naturally when studying the possible generalizations of Perron-Frobenius theory to infinite matrices. Indeed, given a non-negative matrix $A = (A_{i,j})_{i,j \in \mathbb{N}}$ there is a canonical way of associating to A a directed graph Γ with $\Gamma_V = \mathbb{N}$ and an arrow from $i \in \mathbb{N}$ to $j \in \mathbb{N}$ when $A_{i,j} \neq 0$. If we choose F to take the value $\log A_{ij} - \log \lambda$ on that arrow and set $\beta = -1$, the first of the two questions asks for the existence of a non-zero and non-negative solution ψ to the eigenvalue equation

$$A\psi = \lambda\psi,$$

and the second asks for the structure of the solutions. It is well-known that when the matrix A is not finite, or equivalently when the graph Γ is not finite, it is often very hard to answer these questions. But some

general theory is available and for certain special classes of matrices or graphs we do have fairly complete answers.

Formulated as a problem of finding positive eigenvalues and eigenvectors of a (typically) infinite non-negative matrix the two questions arise in many different settings; most notably perhaps in the theory of countable state Markov chains. The authors interest in the questions stems from their relation to the study of equilibrium states in quantum statistical models where the time-evolution is given by a one-parameter group of automorphism on a C^* -algebra associated to Γ and F in a canonical way. The formulation of the two questions are chosen to match the way they come up in that setting, cf. [Th2]. However, it should be clear that an interest in them can come from many different areas of mathematics, and operator algebras will not play any role in this paper. For the significance of harmonic vectors in connection with countable state Markov chains and random walks we refer to the monographs by Woess, [Wo1] and [Wo2].

The purpose here is to show that the questions are tightly connected to the ergodic theory for the shift map on the space of right-infinite paths in Γ . Specifically, we show in the first section below that $A(\beta)$ -harmonic vectors are in bijective correspondence with the regular Borel measures on the path space of the graph Γ that are conformal with respect to the shift map and a potential function defined from β and F . The main result, whose proof occupies the main part of the paper, is that under this bijection the *minimal* $A(\beta)$ -harmonic vectors correspond exactly to the conformal measures that are *ergodic* for the shift. We refer to the survey article by Sarig, [S], for the significance of conformal measures in connection with the shift map and other dynamical systems.

When the graph Γ is strongly connected and the matrix $A(\beta)$ is recurrent in the sense that

$$\sum_{n=0}^{\infty} A(\beta)_{v,v}^n = \infty$$

for one and hence all vertexes v , we know from the work of Vere-Jones, [V], when there is an $A(\beta)$ -harmonic vector and that it is then unique up to scalar multiplication. It is also known that the corresponding conformal measure is conservative and ergodic for the shift, cf. Theorem 3.5 in [S]. We focus therefore in the present paper on the transient case where

$$\sum_{n=0}^{\infty} A(\beta)_{v,w}^n < \infty$$

for all v, w . In this case there may be many linearly independent solutions to (1.2) and the conformal measures in question are all dissipative.

That minimality of an $A(\beta)$ -harmonic vector implies ergodicity of the corresponding conformal measure is the easy part; it is the proof of

the reverse implication which requires substantial work. It is achieved here by showing, in Section 3, that disintegration of conformal measures works well with respect to sufficiently nice shift-invariant maps, and by extending, in Section 4, the Martin kernel approach from the study of harmonic vectors for countable state Markov chains to the more general setting we consider here. For the last purpose we follow some of the way on the route given in the work of Sawyer, [Sa], for his proof of the convergence to the boundary theorem for transient countable state Markov chains. But we avoid to introduce the Martin boundary or any replacement of it. The main result is obtained in the last section as a combination of the results from Sections 3 and 4.

2. CONFORMAL MEASURES AND HARMONIC VECTORS

An element $p = (p_i)_{i=1}^\infty \in (\Gamma_E)^\mathbb{N}$ is an *infinite path* in Γ when $r(p_i) = s(p_{i+1})$ for all i . The set $P(\Gamma)$ of infinite paths in Γ is a complete metric space with metric

$$d(p, q) = \sum_{i=1}^{\infty} 2^{-i} \delta(p_i, q_i),$$

where $\delta : \Gamma_E \times \Gamma_E \rightarrow \{0, 1\}$ is defined such that $\delta(a, b) = 0$ iff $a = b$. We extend the source map s to $P(\Gamma)$ such that $s(p) = s(p_1)$ when $p = (p_i)_{i=1}^\infty \in P(\Gamma)$. A *finite path* μ of length $|\mu| = n$ is an element $\mu = (p_i)_{i=1}^n \in (\Gamma_E)^n$ such that $r(p_i) = s(p_{i+1})$, $i = 1, 2, \dots, n-1$, and we set $s(\mu) = s(p_1)$, $r(\mu) = r(p_n)$. A vertex $v \in \Gamma_V$ is considered as a finite path of length 0. Let $P_f(\Gamma)$ denote the set of all finite paths in Γ . We extend F to a map $F : P_f(\Gamma) \rightarrow \mathbb{R}$ such that $F(v) = 0$ when $v \in \Gamma_V$ and

$$F(\mu) = \sum_{i=1}^n F(p_i)$$

when $\mu = (p_i)_{i=1}^n \in (\Gamma_E)^n$. Associated to the finite path μ is the cylinder set

$$Z(\mu) = \{(x_i)_{i=1}^\infty \in P(\Gamma) \mid x_j = p_j, j = 1, 2, \dots, n\}$$

which is an open and closed set in $P(\Gamma)$. In particular, when μ has length 0, and hence is just a vertex v ,

$$Z(v) = \{p \in P(\Gamma) \mid s(p) = v\}.$$

The *shift map* $\sigma : P(\Gamma) \rightarrow P(\Gamma)$ is defined such that

$$\sigma(p)_i = p_{i+1}.$$

Note that σ is continuous and injective on $Z(a)$ for each $a \in \Gamma_E$.

Definition 2.1. A Borel measure m on $P(\Gamma)$ is $e^{\beta F}$ -conformal when

$$m(\sigma(B \cap Z(a))) = e^{\beta F(a)} m(B \cap Z(a)) \quad (2.1)$$

for every edge $a \in \Gamma_E$ and every Borel subset B of $P(\Gamma)$.

Given a vertex $v \in \Gamma_V$, a finite path μ with $r(\mu) = v$ and a subset $A \subseteq Z(v)$ we let $Z(\mu)A$ denote the set $Z(\mu) \cap \sigma^{-|\mu|}(A)$. The next lemma will be used often and sometimes without reference in the following. The proof is obvious.

Lemma 2.2. *Let m an $e^{\beta F}$ -conformal Borel measure on $P(\Gamma)$. Consider a vertex $v \in \Gamma_V$, a Borel subset $B \subseteq Z(v)$ and a finite path μ in Γ such that $r(\mu) = v$. Then*

$$m(Z(\mu)B) = e^{-\beta F(\mu)}m(B).$$

With a slight abuse of terminology we say that a Borel measure m on $P(\Gamma)$ is *regular* when $m(Z(v)) < \infty$ for all $v \in \Gamma_V$.

Proposition 2.3. *There is a bijection $m \mapsto \psi$ between the set of regular $e^{\beta F}$ -conformal measures m on $P(\Gamma)$ and the $A(\beta)$ -harmonic vectors ψ given by*

$$\psi_v = m(Z(v)), \quad v \in \Gamma_V.$$

Proof. When m is a regular $e^{\beta F}$ -conformal Borel measure on $P(\Gamma)$, the calculation

$$\begin{aligned} \sum_{w \in \Gamma_V} A(\beta)_{v,w} m(Z(w)) &= \sum_{w \in \Gamma_V} \sum_{a \in s^{-1}(v) \cap r^{-1}(w)} e^{-\beta F(a)} m(Z(w)) \\ &= \sum_{w \in \Gamma_V} \sum_{a \in s^{-1}(v) \cap r^{-1}(w)} e^{-\beta F(a)} m(\sigma(Z(a))) \\ &= \sum_{w \in \Gamma_V} \sum_{a \in s^{-1}(v) \cap r^{-1}(w)} m(Z(a)) = m(Z(v)) \end{aligned}$$

shows that the corresponding vector ψ is $A(\beta)$ -harmonic. To show that the map is injective consider two regular $e^{\beta F}$ -conformal measures m and m' . If $m(Z(v)) = m'(Z(v))$ for all $v \in \Gamma_V$ it follows from Lemma 2.2 that $m(Z(\mu)) = m'(Z(\mu))$ for all $\mu \in P_f(\Gamma)$. Since the cylinder sets form a π -system which generates the Borel σ -algebra it follows from standard results in measure theory that m and m' agree on the Borel subsets of $Z(v)$ for any fixed $v \in \Gamma_V$, cf. e.g. Corollary 1.6.2 in [Co]. As $Z(v), v \in \Gamma_V$, is a countable partition of $P(\Gamma)$ into Borel sets it follows that $m = m'$.

The surjectivity of the map follows from standard constructions in measure theory: Let ψ be an $A(\beta)$ -harmonic vector. For each $n \in \mathbb{N}$ the cylinders $Z(\mu)$ with $|\mu| = n$ constitute a countable partition of $P(\Gamma)$ and if we let \mathcal{A}_n denote the σ -algebra of sets defined by this partition we get an increasing sequence $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots$ of σ -algebras whose union

$$\mathcal{A} = \bigcup_n \mathcal{A}_n$$

is an algebra where we can define an additive set function $m : \mathcal{A} \rightarrow [0, \infty)$ such that

$$m(Z(\mu)) = e^{-\beta F(\mu)} \psi_{r(\mu)}.$$

A standard argument, as in the proof of Theorem 1.12 in [Wo2], shows that m is countably additive and hence a measure. By Theorem 1-19 in [KSK] it extends uniquely to a measure m on the σ -algebra generated by the cylinder sets, which is the Borel σ -algebra. To show that the extended measure is $e^{\beta F}$ -conformal consider an edge $a \in \Gamma_E$. The two Borel measures on $P(\Gamma)$ given by

$$B \mapsto m(\sigma(B \cap Z(a)))$$

and

$$B \mapsto e^{\beta F(a)} m(B \cap Z(a)),$$

respectively, agree on all cylinder sets by definition, and the uniqueness part of Theorem 1-19 in [KSK] therefore imply that they are identical (alternatively, one can use Corollary 1.6.2 in [Co] again), i.e. (2.1) holds for every Borel subset B of $P(\Gamma)$. This finishes the proof since $m(Z(v)) = \psi_v$ for all $v \in \Gamma_V$. \square

A non-zero $A(\beta)$ -harmonic vector ψ is *minimal* when every $A(\beta)$ -harmonic vector ϕ with the property that $\phi_v \leq \psi_v$ for all $v \in \Gamma_V$ is a multiple of ψ , i.e. $\phi = \lambda\psi$ for some $\lambda \geq 0$. The aim here is to relate this condition to ergodicity. Recall that a Borel measure m on $P(\Gamma)$ is *ergodic* with respect to the shift σ when every Borel subset $B \subseteq P(\Gamma)$ which is totally shift invariant in the sense that $\sigma^{-1}(B) = B$ is either a null-set or a co-null set for m , i.e. either $m(B) = 0$ or $m(P(\Gamma) \setminus B) = 0$. We shall show that the $A(\beta)$ -harmonic vector of Proposition 2.3 is minimal if and only if the corresponding $e^{\beta F}$ -conformal measure m is ergodic for σ . This will be done under the assumption that there is a vertex v_0 in Γ from where all vertexes can be reached. We shall formulate this condition by requiring

$$\sum_{n=0}^{\infty} A(\beta)_{v_0, v}^n > 0$$

for all $v \in \Gamma_V$. Furthermore, as explained in the introduction we will later assume transience of $A(\beta)$, but in the present section it suffices to assume that

$$A(\beta)_{v_0, v}^n < \infty$$

for all $n \in \mathbb{N}$ and all $v \in \Gamma_V$. For each vertex $v \in \Gamma_V$ we can then choose $k \in \mathbb{N}$ such that $A(\beta)_{v_0, v}^k > 0$ and set

$$b_v = (A(\beta)_{v_0, v}^k)^{-1}.$$

A vector $\psi : \Gamma_V \rightarrow [0, \infty)$ is *normalized* when $\psi_{v_0} = 1$ and a regular measure m on $P(\Gamma)$ is *normalized* when $m(Z(v_0)) = 1$. When ψ is a normalized $A(\beta)$ -harmonic vector the estimate

$$A(\beta)_{v_0, v}^k \psi_v \leq \sum_{w \in \Gamma_V} A(\beta)_{v_0, w}^k \psi_w = \psi_{v_0} = 1$$

shows that $\psi_v \leq b_v$. The set of normalized $A(\beta)$ -harmonic vectors is therefore a convex subset of the compact product space

$$\prod_{v \in \Gamma_V} [0, b_v].$$

Let V_∞ be set of vertexes in Γ that emit infinitely many arrows, i.e.

$$V_\infty = \{v \in \Gamma_V : \#s^{-1}(v) = \infty\},$$

and let Δ be the convex and compact set subset of $\prod_{v \in \Gamma_V} [0, b_v]$ consisting of the elements ψ which satisfy that

- i) $\sum_{w \in \Gamma_V} A(\beta)_{v,w} \psi_w \leq \psi_v \quad \forall v \in \Gamma_V$,
- ii) $\sum_{w \in \Gamma_V} A(\beta)_{v,w} \psi_w = \psi_v \quad \forall v \in \Gamma_V \setminus V_\infty$, and
- iii) $\psi_{v_0} = 1$.

The set of normalized $A(\beta)$ -harmonic vectors is a (possibly empty) convex face in Δ . A non-negative vector ψ for which i) holds is called $A(\beta)$ -superharmonic.

Let $\partial\Delta$ be the set of extreme points in Δ ; a Borel subset by Choquet theory. In fact, $\partial\Delta$ is a G_δ -set by Theorem 4.1.11 in [BR]. Set

$$\partial'\Delta = \left\{ \psi \in \partial\Delta : \sum_{w \in \Gamma_V} A(\beta)_{v,w} \psi_w = \psi_v \quad \forall v \in \Gamma_V \right\},$$

which is the set of normalized minimal $A(\beta)$ -harmonic vectors. Note that it is a Borel subset of $\partial\Delta$.

A non-zero regular $e^{\beta F}$ -conformal measure m is *extremal* when all regular $e^{\beta F}$ -conformal measures m' such that $m' \leq m$ are scalar multiples of m , i.e. $m' = \lambda m$ for some $\lambda \geq 0$. For normalized $e^{\beta F}$ -conformal measures this is the same as being extremal in the set of normalized $e^{\beta F}$ -conformal measures. Clearly, an $A(\beta)$ -harmonic vector is minimal if and only if the corresponding regular $e^{\beta F}$ -conformal measure is extremal. Thus the following is an immediate consequence of Proposition 2.3.

Lemma 2.4. *For each $x \in \partial'\Delta$ there is a unique extremal normalized $e^{\beta F}$ -conformal measure m_x on $P(\Gamma)$ such that*

$$m_x(Z(v)) = x_v$$

for all $v \in \Gamma_V$.

Note that the map $x \mapsto m_x(Z(v))$ is continuous on $\partial'\Delta$ for all $v \in \Gamma_V$, and that consequently $x \mapsto m_x(B)$ is a Borel map on $\partial'\Delta$ for all Borel subsets $B \subseteq P(\Gamma)$. Hence every Borel probability measure ν on $\partial'\Delta$ gives rise to a measure

$$\int_{\partial'\Delta} m_x \, d\nu(x)$$

on $P(\Gamma)$ defined such that

$$B \mapsto \int_{\partial'\Delta} m_x(B) \, d\nu(x)$$

Note that $\int_{\partial'\Delta} m_x \, d\nu(x)$ is normalized and $e^{\beta F}$ -conformal since each m_x is.

Proposition 2.5. *Let m be a normalized $e^{\beta F}$ -conformal measure on $P(\Gamma)$. There is a Borel probability measure ν on $\partial'\Delta$ such that*

$$m = \int_{\partial'\Delta} m_x \, d\nu(x). \quad (2.2)$$

Proof. It follows from Proposition 2.3 and Choquet theory that there is a Borel probability measure ν on $\partial\Delta$ such that

$$m(Z(v)) = \int_{\partial\Delta} x_v \, d\nu(x)$$

for all $v \in \Gamma_V$, cf. e.g. Proposition 4.1.3 and Theorem 4.1.11 in [BR]. Since

$$\begin{aligned} & \int_{\partial\Delta} \left(x_v - \sum_{w \in \Gamma_V} A(\beta)_{v,w} x_w \right) \, d\nu(x) \\ &= m(Z(v)) - \sum_{w \in \Gamma_V} A(\beta)_{v,w} m(Z(w)) = 0, \end{aligned}$$

for all $v \in \Gamma_V$, it follows that ν is concentrated on $\partial'\Delta$. The equality (2.2) follows from Proposition 2.3 since

$$\int_{\partial'\Delta} m_x(Z(v)) \, d\nu(x) = \int_{\partial'\Delta} x_v \, d\nu(x) = m(Z(v)).$$

□

Although we shall not need the fact, it should be noted that Δ is a Choquet simplex, cf. Theorem 4.6 in [Th1], which implies that the representing measure ν is unique in Proposition 2.5. Thus the convex set of normalized $e^{\beta F}$ -conformal measures on $P(\Gamma)$ is in affine bijection with the set of Borel probability measures on $\partial'\Delta$.

3. DISINTEGRATION OF CONFORMAL MEASURES

Lemma 3.1. *Let X be a second countable topological Hausdorff space and $T : P(\Gamma) \rightarrow X$ a Borel map. Let m be a regular Borel measure on $P(\Gamma)$ and μ a σ -finite Borel measure on X such that $m \circ T^{-1}$ is absolutely continuous with respect to μ (i.e. $\mu(B) = 0 \Rightarrow m(T^{-1}(B)) = 0$). Then m has a (T, μ) -integration, i.e. there are regular Borel measures $m_x, x \in X$, on $P(\Gamma)$ such that*

a) m_x is concentrated on $T^{-1}(x)$ for μ -almost all $x \in X$,
and for every Borel function $f : P(\Gamma) \rightarrow [0, \infty]$,

- b) the function $X \ni x \mapsto \int_{T^{-1}(x)} f(y) \, dm_x(y)$ is Borel, and
 c)

$$\int_{P(\Gamma)} f \, dm = \int_X \left(\int_{T^{-1}(x)} f(y) \, dm_x(y) \right) d\mu(x).$$

The disintegration is unique in the following sense: If $m'_x, x \in X$, is another collection of regular Borel measures on $P(\Gamma)$ for which a), b) and c) hold, then $m'_x = m_x$ for μ -almost all $x \in X$.

Proof. This follows from Theorem 1 in [CP]. We only have to observe that m is σ -finite and finite on compact sets and inner regular since this is part of the assumptions in [CP]. The first two properties follow from the assumed finiteness of $m(Z(v))$ since $Z(v), v \in \Gamma_V$, is an open cover of $P(\Gamma)$. Let B be a Borel subset of $P(\Gamma)$ and let $\epsilon > 0$. Since $Z(v)$ is a Polish space in the relative topology inherited from $P(\Gamma)$ the measure $A \mapsto m(A \cap Z(v))$ is inner regular for all $v \in V$ by Proposition 8.1.10 in [Co] and there is therefore a compact subset $K_v \subseteq Z(v) \cap B$ such that $m(K_v) \geq (1 - \epsilon)m(B \cap Z(v))$. Then

$$m\left(\bigcup_{v \in \Gamma_V} K_v\right) = \sum_{v \in \Gamma_V} m(K_v) \geq (1 - \epsilon)m(B),$$

so for some sufficiently large finite subset $F \subseteq \Gamma_V$ we have that $K = \bigcup_{v \in F} K_v$ is a compact subset of B such that $m(K) \geq (1 - 2\epsilon)m(B)$. Hence m is inner regular. \square

Theorem 3.2. *Let m be a non-zero regular $e^{\beta F}$ -conformal measure on $P(\Gamma)$. Let X be a second countable topological Hausdorff space and $T : P(\Gamma) \rightarrow X$ a Borel map. Assume that $T(\sigma(p)) = T(p)$ for all $p \in P(\Gamma)$.*

- i) *There is a Borel probability measure on X with the same null-sets as the pushforward measure $m \circ T^{-1}$.*
- ii) *For every σ -finite Borel measure ν on X with $m \circ T^{-1}$ absolutely continuous with respect to ν there is a (T, ν) -integration $m_x, x \in X$, of m such that m_x is a regular $e^{\beta F}$ -conformal measure for ν -almost every $x \in X$.*

Proof. i) Set $M = \{v \in \Gamma_V : m(Z(v)) \neq 0\}$. Note that $M \neq \emptyset$ because $m \neq 0$. Choose positive real numbers λ_v such that $\sum_{v \in M} \lambda_v = 1$, and define a Borel probability measure m' on $P(\Gamma)$ such that

$$m'(B) = \sum_{v \in M} \lambda_v \frac{m(Z(v) \cap B)}{m(Z(v))}.$$

Note that m has the same null-sets as m' . It follows that the pushforward measure $m \circ T^{-1}$ has the same null-sets as the Borel probability measure $m' \circ T^{-1}$.

ii) By Lemma 3.1 there is a (T, ν) -integration $m_x, x \in X$, of m . Consider an edge $a \in \Gamma_E$. Define Borel measures m^a and n^a on $P(\Gamma)$ such that

$$n^a(B) = m(\sigma(B \cap Z(a))) = e^{\beta F(a)} m(B \cap Z(a))$$

and

$$m^a(B) = m(B \cap Z(a)),$$

respectively. For each $x \in X$ define Borel measures m_x^a and n_x^a on $P(\Gamma)$ by

$$n_x^a(B) = m_x(\sigma(B \cap Z(a)))$$

and

$$m_x^a(B) = m_x(B \cap Z(a)),$$

respectively. When f is a positive Borel function on $P(\Gamma)$ we have that

$$\begin{aligned} \int_{P(\Gamma)} f \, dm^a &= \int_{P(\Gamma)} 1_{Z(a)} f \, dm \\ &= \int_X \left(\int_{T^{-1}(x)} 1_{Z(a)} f \, dm_x \right) d\nu(x) = \int_X \left(\int_{T^{-1}(x)} f \, dm_x^a \right) d\nu(x). \end{aligned}$$

Since $n^a = e^{\beta F(a)} m^a$ it follows that

$$\int_{P(\Gamma)} f \, dn^a = \int_X \left(\int_{T^{-1}(x)} f \, e^{\beta F(a)} dm_x^a \right) d\nu(x). \quad (3.1)$$

The transformation theorem for integrals shows that

$$\int_{T^{-1}(x)} f \, dn_x^a = \int_{T^{-1}(x)} g \, dm_x$$

where $g : P(\Gamma) \rightarrow [0, \infty)$ is the Borel function

$$g(p) = \begin{cases} 0, & p \notin Z(r(a)) \\ f(\sigma^{-1}(p)), & p \in Z(r(a)), \end{cases}$$

and $\sigma^{-1} : Z(r(a)) \rightarrow Z(a)$ is the inverse of $\sigma : Z(a) \rightarrow Z(r(a))$. It follows that the map $x \mapsto \int_{T^{-1}(x)} f \, dn_x^a$ is Borel. Let $B \subseteq P(\Gamma)$ be a Borel set. Then

$$\int_X \left(\int_{T^{-1}(x)} 1_B \, dn_x^a \right) d\nu(x) = \int_X m_x(\sigma(B \cap Z(a) \cap T^{-1}(x))) \, d\nu(x).$$

Since $T \circ \sigma = T$ by assumption, it follows that

$$\sigma(B \cap Z(a) \cap T^{-1}(x)) = \sigma(B \cap Z(a)) \cap T^{-1}(x)$$

and hence

$$\begin{aligned} \int_X \left(\int_{T^{-1}(x)} 1_B \, dn_x^a \right) d\nu(x) &= \int_X m_x(\sigma(B \cap Z(a))) \, d\nu(x) \\ &= m(\sigma(B \cap Z(a))) = e^{\beta F(a)} m(B \cap Z(a)) = n^a(B). \end{aligned}$$

Since B was arbitrary it follows that

$$\int_X \left(\int_{T^{-1}(x)} f \, dn_x^a \right) d\nu(x) = \int_{P(\Gamma)} f \, dn^a. \quad (3.2)$$

Note that n_x^a is concentrated on $T^{-1}(x)$ since $\sigma^{-1}(T^{-1}(x)) \subseteq T^{-1}(x)$ by assumption. Hence n_x^a and $e^{\beta F(a)} m_x^a$ are both regular Borel measures concentrated on $T^{-1}(x)$ and as (3.1) and (3.2) show, they both define (T, ν) -disintegrations of n^a . It follows therefore from the essential uniqueness of the disintegration, cf. Lemma 3.1, that $n_x^a = e^{\beta F(a)} m_x^a$ for ν -almost every x . That is, for ν -almost all x we have that

$$m_x(\sigma(B \cap Z(a))) = e^{\beta F(a)} m_x(B \cap Z(a))$$

for all Borel sets B in $P(\Gamma)$. It follows that m_x is $e^{\beta F}$ -conformal for ν -almost all x . \square

4. CONFORMAL MEASURES AND MARTIN KERNELS

This part is heavily inspired by the work of Sawyer, [Sa]. It is a standing assumption that there is a vertex v_0 such that

$$0 < \sum_{n=0}^{\infty} A(\beta)_{v_0, v}^n < \infty$$

for all $v \in \Gamma_V$.

Lemma 4.1. *Let m be a regular $e^{\beta F}$ -conformal measure on $P(\Gamma)$, and let $M \subseteq \Gamma_V$ be any set of vertexes. Then*

$$\varphi_v = m(\{p \in Z(v) : s(p_k) \in M \text{ for infinitely many } k\})$$

defines an $A(\beta)$ -harmonic vector φ .

Proof. It follows from Lemma 2.2 that

$$\begin{aligned} & A(\beta)_{v, w} \varphi_w \\ &= \sum_{a \in s^{-1}(v) \cap r^{-1}(w)} m(\{p \in Z(a) : s(p_k) \in M \text{ for infinitely many } k\}) \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{w \in \Gamma_V} A(\beta)_{v, w} \varphi_w \\ &= \sum_{a \in s^{-1}(v)} m(\{p \in Z(a) : s(p_k) \in M \text{ for infinitely many } k\}) \\ &= m(\{p \in Z(v) : s(p_k) \in M \text{ for infinitely many } k\}) = \varphi_v. \end{aligned}$$

\square

Lemma 4.2. *Let m be an extremal regular $e^{\beta F}$ -conformal measure on $P(\Gamma)$, and let $M \subseteq \Gamma_V$ be any set of vertexes. Then either*

$$m(\{p \in Z(v) : s(p_k) \in M \text{ for infinitely many } k\}) = 0$$

for all $v \in \Gamma_V$, or

$$m(\{p \in Z(v) : s(p_k) \in M \text{ for infinitely many } k\}) = m(Z(v))$$

for all $v \in \Gamma_V$.

Proof. By extremality and Lemma 4.1 there is a $\lambda \in [0, 1]$ such that

$$m(\{p \in Z(v) : s(p_k) \in M \text{ for infinitely many } k\}) = \lambda m(Z(v))$$

for all $v \in \Gamma_V$. Set $E = \{p \in P(\Gamma) : s(p_k) \in M \text{ for infinitely many } k\}$. Fix $v \in \Gamma_V$ and consider a finite path μ with $s(\mu) = v$. Then $Z(\mu) \cap E = Z(\mu)(Z(r(\mu)) \cap E)$, and hence Lemma 2.2 implies that

$$\begin{aligned} m(Z(\mu) \cap E) &= e^{-\beta F(\mu)} m(Z(r(\mu)) \cap E) \\ &= e^{-\beta F(\mu)} \lambda m(Z(r(\mu))) = \lambda m(Z(\mu)) \end{aligned} \quad (4.1)$$

Let $\epsilon > 0$ and consider a vertex $v \in \Gamma_V$. Since m is a finite measure on $Z(v)$ and the Borel σ -algebra is generated by the cylinder sets it follows that there is a finite collection of mutually disjoint cylinders $Z(\mu_i)$, all with $s(\mu_i) = v$, such that

$$m\left((E \cap Z(v)) \setminus \left(\bigcup_i Z(\mu_i)\right)\right) + m\left(\left(\bigcup_i Z(\mu_i)\right) \setminus (Z(v) \cap E)\right) \leq \epsilon,$$

cf. e.g. [KT], p. 84. Applying (4.1) to each μ_i we find that

$$\begin{aligned} m(Z(v) \cap E) - \epsilon &\leq m\left(Z(v) \cap E \cap \left(\bigcup_i Z(\mu_i)\right)\right) \\ &= \sum_i m(Z(\mu_i) \cap E) = \lambda \sum_i m(Z(\mu_i)) \\ &\leq \lambda m\left(E \cap Z(v) \cap \left(\bigcup_i Z(\mu_i)\right)\right) + \lambda \epsilon \\ &\leq \lambda m(Z(v) \cap E) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary we conclude that $m(Z(v) \cap E) = \lambda m(Z(v) \cap E)$. And $v \in \Gamma_V$ was also arbitrary, so it follows that $\lambda = 1$ unless $m(E) = 0$ in which case $\lambda = 0$. \square

For each $v \in \Gamma_V$ we choose $k \in \mathbb{N}$ such that $A(\beta)_{v_0, v}^k \neq 0$. Then

$$A(\beta)_{v_0, v}^k \sum_{n=0}^{\infty} A(\beta)_{v, w}^n \leq \sum_{n=0}^{\infty} A(\beta)_{v_0, w}^{n+k} \leq \sum_{n=0}^{\infty} A(\beta)_{v_0, w}^n \quad (4.2)$$

for all $v, w \in \Gamma_V$. In particular, $\sum_{n=0}^{\infty} A(\beta)_{v,w}^n < \infty$ for all $v, w \in \Gamma_V$ and we can define the *Martin kernel* $K_\beta : \Gamma_V \times \Gamma_V \rightarrow [0, \infty)$ such that

$$K_\beta(v, w) = \frac{\sum_{n=0}^{\infty} A(\beta)_{v,w}^n}{\sum_{n=0}^{\infty} A(\beta)_{v_0,w}^n}.$$

It follows from (4.2) that $K_\beta(v, w) \leq b_v$ where $b_v = 1/A(\beta)_{v_0,v}^k$. We define

$$K_\beta : \Gamma_V \rightarrow \prod_{v \in \Gamma_V} [0, b_v]$$

such that $K_\beta(w) = (K_\beta(v, w))_{v \in \Gamma_V}$. Set

$$\partial K_\beta = \overline{K_\beta(\Gamma_V)} \setminus K_\beta(\Gamma_V).$$

It is straightforward to check that $\partial K_\beta \subseteq \Delta$; the compact convex set used in Section 2. Since

$$\sum_{w \in \Gamma_V} A(\beta)_{v,w}^{n+1} \xi_w \leq \sum_{w \in V} A(\beta)_{v,w}^n \xi_w$$

for all n, v when $\xi \in \overline{K_\beta(\Gamma_V)}$, we obtain a Borel function $q_v : \overline{K_\beta(\Gamma_V)} \rightarrow [0, b_v]$ for every $v \in \Gamma_V$ defined such that

$$q_v(\xi) = \lim_{n \rightarrow \infty} \sum_{w \in \Gamma_V} A(\beta)_{v,w}^n \xi_w.$$

Lemma 4.3. *Let m be a normalized $e^{\beta F}$ -conformal measure. Let $M \subseteq \Gamma_V$ be a set of vertexes containing v_0 . There is a Borel probability measure μ_M on $\overline{K_\beta(M)}$ such that*

$$\begin{aligned} & m(\{p \in Z(v) : s(p_k) \in M \text{ for infinitely many } k\}) \\ &= \int_{\overline{K_\beta(M)}} q_v(\xi) \, d\mu_M(\xi) \end{aligned} \tag{4.3}$$

for all $v \in \Gamma_V$.

Proof. Consider a finite subset $F \subseteq M$ containing v_0 . Using Lemma 2.2 we find that

$$\begin{aligned} & \sum_{w \in \Gamma_V} A(\beta)_{v,w}^n m(\{p \in Z(w) : s(p_k) \in F \text{ for some } k \geq 1\}) \\ &= m(\{p \in Z(v) : s(p_k) \in F \text{ for some } k \geq n+1\}) \\ &\leq m(\{p \in Z(v) : s(p_k) \in F \text{ for some } k \geq 1\}). \end{aligned} \tag{4.4}$$

The case $n = 1$ shows that the vector φ given by

$$\varphi_v = m(\{p \in Z(v) : s(p_k) \in F \text{ for some } k \geq 1\})$$

is $A(\beta)$ -superharmonic. Note that

$$\{p \in Z(v) : s(p_k) \in F \text{ for some } k \geq n\} \subseteq \bigcup_{\mu \in L} Z(\mu),$$

where

$$L = \{\mu \in P_f(\Gamma) : s(\mu) = v, r(\mu) \in F, |\mu| \geq n-1\}.$$

Hence

$$m(\{p \in Z(v) : s(p_k) \in F \text{ for some } k \geq n\}) \leq \sum_{w \in F} \sum_{k \geq n-1} A(\beta)_{v,w}^k,$$

which implies that

$$\lim_{n \rightarrow \infty} m(\{p \in Z(v) : s(p_k) \in F \text{ for some } k \geq n\}) = 0. \quad (4.5)$$

Using this in (4.4) we find that

$$\lim_{n \rightarrow \infty} \sum_{w \in \Gamma_V} A(\beta)_{v,w}^n \varphi_w = 0. \quad (4.6)$$

Set

$$k_F(w) = \varphi_w - \sum_{u \in \Gamma_V} A(\beta)_{w,u} \varphi_u.$$

Since

$$\sum_{u \in \Gamma_V} A(\beta)_{w,u} \varphi_u = m(\{p \in Z(w) : s(p_k) \in F \text{ for some } k \geq 2\}),$$

we see that

$$k_F(w) = m(\{p \in Z(w) : s(p_1) \in F, s(p_k) \notin F \forall k \geq 2\}).$$

In particular, k_F is supported in F . Furthermore,

$$\begin{aligned} \sum_{w \in F} \sum_{n=0}^{\infty} A(\beta)_{v,w}^n k_F(w) &= \sum_{w \in \Gamma_V} \sum_{n=0}^{\infty} A(\beta)_{v,w}^n k_F(w) \\ &= \lim_{N \rightarrow \infty} \sum_{n=0}^N \sum_{w \in \Gamma_V} A(\beta)_{v,w}^n k_F(w) \\ &= \lim_{N \rightarrow \infty} \sum_{n=0}^N \left(\sum_{w \in \Gamma_V} A(\beta)_{v,w}^n \varphi_w - \sum_{w \in \Gamma_V} \sum_{u \in \Gamma_V} A(\beta)_{v,w}^n A(\beta)_{w,u} \varphi_u \right) \quad (4.7) \\ &= \lim_{N \rightarrow \infty} \sum_{n=0}^N \left(\sum_{w \in \Gamma_V} A(\beta)_{v,w}^n \varphi_w - \sum_{u \in \Gamma_V} A(\beta)_{v,u}^{n+1} \varphi_u \right) \\ &= \lim_{N \rightarrow \infty} \left(\varphi_v - \sum_{u \in \Gamma_V} A(\beta)^{N+1} \varphi_u \right) = \varphi_v, \end{aligned}$$

where we used (4.6) in the last step. It follows from (4.7) that

$$\varphi_v = \sum_{w \in F} K_\beta(v, w) \sum_{n=0}^{\infty} A(\beta)_{v_0,w}^n k_F(w). \quad (4.8)$$

Note that

$$\begin{aligned}
& \sum_{w \in F} \sum_{n=0}^{\infty} A(\beta)_{v_0, w}^n k_F(w) \\
&= \sum_{w \in F} \sum_{n=0}^{\infty} A(\beta)_{v_0, w}^n m(\{p \in Z(w) : s(p_k) \notin F \ \forall k \geq 2\}) \\
&= m(\{p \in Z(v_0) : s(p_k) \notin F \text{ for all large enough } k\}) \\
&= m(Z(v_0)) = 1.
\end{aligned}$$

where the second to last equality follows from (4.5). Hence (4.8) can be written

$$\varphi_v = \int_{\Gamma_V} K_\beta(v, w) \, d\nu(w),$$

where ν is the probability measure

$$\nu = \sum_{w \in F} \sum_{n=0}^{\infty} A(\beta)_{v_0, w}^n k_F(w) \delta_w.$$

We obtain in this way the push-forward measure $\mu_F = \nu \circ K_\beta^{-1}$ on $\overline{K_\beta(\Gamma_V)}$ which is supported on $K_\beta(F)$. Note that

$$\varphi_v = \int_{\overline{K_\beta(V)}} \xi_v \, d\mu_F(\xi).$$

Now choose a sequence $v_0 \in F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots$ of finite subsets of M such that $\bigcup_i F_i = M$. By the preceding we have for each i a Borel probability measure μ_i supported in $\overline{K_\beta(M)}$ such that

$$m(\{p \in Z(v) : s(p_k) \in F_i \text{ for some } k\}) = \int_{\overline{K_\beta(M)}} \xi_v \, d\mu_i(\xi)$$

for all $v \in \Gamma_V$. Let μ_M be a weak*-condensation point of the sequence $\{\mu_i\}$ in the set of Borel probability measures on $\overline{K_\beta(M)}$. Then

$$m(\{p \in Z(v) : s(p_k) \in M \text{ for some } k\}) = \int_{\overline{K_\beta(M)}} \xi_v \, d\mu_M(\xi)$$

for all $v \in \Gamma_V$. Furthermore, we find by using Lemma 2.2 that

$$\begin{aligned}
& m(\{p \in Z(v) : s(p_k) \in M \text{ for some } k \geq n+1\}) \\
&= \sum_{w \in \Gamma_V} A(\beta)_{v, w}^n m(\{p \in Z(w) : s(p_k) \in M \text{ for some } k\}) \\
&= \int_{\overline{K_\beta(M)}} \sum_{w \in \Gamma_V} A(\beta)_{v, w}^n \xi_w \, d\mu_M(\xi).
\end{aligned}$$

Letting n tend to infinity we obtain (4.3). □

Lemma 4.4. *Let m be an extremal normalized $e^{\beta F}$ -conformal measure. Let $M \subseteq \Gamma_V$ be a set of vertexes containing v_0 such that*

$$m(\{p \in Z(v) : s(p_k) \in M \text{ for infinitely many } k\}) = m(Z(v))$$

for all $v \in \Gamma_V$. There is an element $\xi \in \partial K_\beta \cap \overline{K_\beta(M)}$ such that

$$m(Z(v)) = \xi_v$$

for all $v \in \Gamma_V$.

Proof. By Lemma 4.3 there is a Borel probability measure ν on $\overline{K_\beta(M)}$ such that

$$m(Z(v)) = \int_{\overline{K_\beta(M)}} q_v(\xi) \, d\nu(\xi)$$

for all $v \in \Gamma_V$. Note that $q_{v_0}(\xi) \leq \xi_{v_0} = 1$ and that

$$\int_{\overline{K_\beta(M)}} q_{v_0}(\xi) \, d\nu(\xi) = m(Z(v_0)) = 1. \quad (4.9)$$

It follows from this that $q_{v_0}(\xi) = 1$ for ν -almost all ξ . Furthermore,

$$\begin{aligned} q_{v_0}(K_\beta(w)) &= \left(\sum_{n=0}^{\infty} A(\beta)_{v_0,w}^n \right)^{-1} \lim_{k \rightarrow \infty} \sum_{v \in \Gamma_V} A(\beta)_{v_0,v}^k \sum_{n=0}^{\infty} A(\beta)_{v,w}^n \\ &= \left(\sum_{n=0}^{\infty} A(\beta)_{v_0,w}^n \right)^{-1} \lim_{k \rightarrow \infty} \sum_{n \geq k} A(\beta)_{v_0,w}^n = 0 \end{aligned}$$

for all $w \in \Gamma_V$, which shows that q_{v_0} annihilates $K_\beta(\Gamma_V)$. Hence (4.9) also implies that ν is concentrated on $\partial K_\beta \cap \overline{K_\beta(M)}$. Set

$$\xi'_v = \xi_v - q_v(\xi).$$

Then ξ' is $A(\beta)$ -superharmonic and $\xi'_{v_0} = 0$ for ν -almost all $\xi \in \partial K_\beta \cap \overline{K_\beta(M)}$. For each $v \in \Gamma_V$ there is a $k \in \mathbb{N}$ such that $A(\beta)_{v_0,v}^k \neq 0$. Since

$$0 \leq A(\beta)_{v_0,v}^k \xi'_v \leq \sum_{w \in \Gamma_V} A(\beta)_{v_0,w}^k \xi'_w \leq \xi'_{v_0},$$

it follows that for ν -almost all ξ the equality $\xi'_v = 0$ holds for all $v \in \Gamma_V$. Hence

$$m(Z(v)) = \int_{\partial K_\beta \cap \overline{K_\beta(M)}} \xi_v \, d\nu(\xi)$$

Since $(m(Z(v)))_{v \in \Gamma_V}$ is $A(\beta)$ -harmonic we find that

$$\int_{\partial K_\beta \cap \overline{K_\beta(M)}} \xi_v \, d\nu(\xi) = \int_{\partial K_\beta \cap \overline{K_\beta(M)}} \sum_{w \in \Gamma_V} A(\beta)_{v,w} \xi_w \, d\nu(\xi)$$

for all $v \in \Gamma_V$ and it follows from this that ν -almost all elements of $\partial K_\beta \cap \overline{K_\beta(M)}$ are $A(\beta)$ -harmonic. Let B a Borel subset of $\partial K_\beta \cap \overline{K_\beta(M)}$

and assume that $0 < \nu(B) < 1$. We can then define two normalized $A(\beta)$ -harmonic vectors φ and φ' such that

$$\varphi_v = \frac{1}{\nu(B)} \int_B \xi_v \, d\nu(\xi)$$

and

$$\varphi'_v = \frac{1}{1 - \nu(B)} \int_{(\partial K_\beta \cap \overline{K_\beta(M)}) \setminus B} \xi_v \, d\nu(\xi) .$$

Since $m(Z(v)) = \nu(B)\varphi_v + (1 - \nu(B))\varphi'_v$ the assumption that m is extremal implies that

$$\nu(B)m(Z(v)) = \int_B \xi_v \, d\nu(\xi)$$

for all $v \in \Gamma_V$. This identity is obvious when $\nu(B) \in \{0, 1\}$ and hence it holds for all Borel subsets B of $\partial K_\beta \cap \overline{K_\beta(M)}$. It follows from this that the set

$$\left\{ \xi \in \partial K_\beta \cap \overline{K_\beta(M)} : \xi_v = m(Z(v)) \, \forall v \in \Gamma_V \right\}$$

has ν -measure 1; in particular, it is not empty. \square

Theorem 4.5. *Assume that there is a vertex v_0 such that*

$$0 < \sum_{n=0}^{\infty} A(\beta)_{v_0, v}^n < \infty$$

for all $v \in \Gamma_V$. Let m be an extremal normalized $e^{\beta F}$ -conformal measure. For m -almost all $p \in P(\Gamma)$,

$$\lim_{k \rightarrow \infty} K_\beta(v, s(p_k)) = m(Z(v))$$

for all $v \in V$.

Proof. By Lemma 4.4 applied with $M = \Gamma_V$ there is an element $\xi \in \partial K_\beta$ such that $m(Z(v)) = \xi_v$ for all $v \in \Gamma_V$. Fix $v' \in \Gamma_V$ and let $\epsilon > 0$. Set

$$M = \{w \in \Gamma_V : |K_\beta(v', w) - \xi_{v'}| \geq \epsilon\} .$$

Assume for a contradiction that

$$m(\{p \in Z(v) : s(p_k) \in M \text{ for infinitely many } k\}) \neq 0$$

for some $v \in \Gamma_V$. Then

$$m(\{p \in Z(v) : s(p_k) \in M \text{ for infinitely many } k\}) = m(Z(v))$$

for all v by Lemma 4.2 and it follows from Lemma 4.4 that there is a $\xi' \in \partial K_\beta \cap \overline{K_\beta(M)}$ such that $m(Z(v)) = \xi'_v$ for all v . But $|\xi'_{v'} - \xi_{v'}| \geq \epsilon$ since $\xi' \in \overline{K_\beta(M)}$, which gives us the desired contradiction. Thus

$$m(\{p \in Z(v) : s(p_k) \in M \text{ for infinitely many } k\}) = 0$$

for all $v \in \Gamma_V$. Since $\epsilon > 0$ and $v' \in \Gamma_V$ were arbitrary it follows that for m -almost all p it holds that $\lim_{k \rightarrow \infty} K_\beta(v, s(p_k)) = m(Z(v))$ for all $v \in \Gamma_V$. \square

By combining Theorem 4.5 with Proposition 2.5 it is straightforward to obtain the following. We leave the details to the reader.

Corollary 4.6. *Assume that there is a vertex v_0 such that*

$$0 < \sum_{n=0}^{\infty} A(\beta)_{v_0, v}^n < \infty$$

for all $v \in \Gamma_V$. Let m be a regular $e^{\beta F}$ -conformal measure. For m -almost all $p \in P(\Gamma)$ the limit $\lim_{k \rightarrow \infty} K_\beta(v, s(p_k))$ exists, and

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{Z(v_0)} K_\beta(v, s(p_k)) \, dm(p) &= \int_{Z(v_0)} \lim_{k \rightarrow \infty} K_\beta(v, s(p_k)) \, dm(p) \\ &= m(Z(v)) \end{aligned}$$

for all $v \in \Gamma_V$.

5. THE MAIN RESULT

Theorem 5.1. *Assume that there is a vertex v_0 such that*

$$0 < \sum_{n=0}^{\infty} A(\beta)_{v_0, v}^n < \infty$$

for all $v \in \Gamma_V$. Let m be a non-zero regular $e^{\beta F}$ -conformal measure on $P(\Gamma)$. The following are equivalent:

a) *m is concentrated on*

$$\left\{ p \in P(\Gamma) : \lim_{k \rightarrow \infty} K_\beta(v, s(p_k)) = \frac{m(Z(v))}{m(Z(v_0))} \, \forall v \in \Gamma_V \right\}. \quad (5.1)$$

b) *m is extremal among the $e^{\beta F}$ -conformal measures on $P(\Gamma)$.*

c) *m is ergodic for the shift on $P(\Gamma)$.*

Proof. a) \Rightarrow b): Let m' be a regular $e^{\beta F}$ -conformal measure such that $m' \leq m$. Since m is concentrated on (5.1) the same holds for m' and it follows from Corollary 4.6 that

$$m'(Z(v)) = \int_{Z(v_0)} \lim_{k \rightarrow \infty} K_\beta(v, s(p_k)) \, dm'(p) = \frac{m'(Z(v_0))}{m(Z(v_0))} m(Z(v))$$

for all $v \in \Gamma_V$. Hence $m' = \frac{m'(Z(v_0))}{m(Z(v_0))} m$ by Proposition 2.3.

b) \Rightarrow c): Let $A \subseteq P(\Gamma)$ be a Borel subset such that $\sigma^{-1}(A) = A$. If $m(A) \neq 0$ and $m(P(\Gamma) \setminus A) \neq 0$, the measure

$$m'(B) = m(B \cap A)$$

will be a regular $e^{\beta F}$ -conformal measure such that $m' \leq m$, and no $\lambda \geq 0$ will satisfy that $m' = \lambda m$.

c) \Rightarrow a) : We may assume that m is normalized. Set

$$\mathcal{B} = \left\{ (p_i)_{i=1}^\infty \in P(\Gamma) : \lim_{k \rightarrow \infty} K_\beta(v, s(p_k)) \text{ exists for all } v \in \Gamma_V \right\};$$

a Borel subset of $P(\Gamma)$. Add to Δ a point \clubsuit and define $K : P(\Gamma) \rightarrow \Delta \sqcup \{\clubsuit\}$ such that

$$K(p) = \begin{cases} (\lim_{k \rightarrow \infty} K_\beta(v, s(p_k)))_{v \in \Gamma_V}, & p \in \mathcal{B}, \\ \clubsuit, & p \in P(\Gamma) \setminus \mathcal{B}. \end{cases}$$

By Theorem 3.2 there is a Borel probability measure ν on $\Delta \sqcup \{\clubsuit\}$ with the same null-sets as $m \circ K^{-1}$ and there is a (K, ν) -disintegration $m_x, x \in \Delta \sqcup \{\clubsuit\}$, of m . It follows from Corollary 4.6 that $m \circ K^{-1}(\{\clubsuit\}) = 0$, and hence that ν is concentrated on Δ . For any Borel subset $B \subseteq \Delta$, the set $K^{-1}(B)$ is a Borel subset of $P(\Gamma)$ which is totally shift invariant; viz. $\sigma^{-1}(K^{-1}(B)) = K^{-1}(B)$. Since m is ergodic by assumption and since ν has the same null-sets as $m \circ K^{-1}$ it follows that $\nu(B) = 0$ or $\nu(\Delta \setminus B) = 0$.

Since Δ is second countable and Hausdorff there is a sequence

$$P^1, P^2, P^3, \dots$$

of finite Borel partitions of Δ such that for any sequence $A_i, i \in \mathbb{N}$, of Borel sets with A_i an element of P^i , the intersection $\bigcap_i A_i$ contains at most a single point from Δ . It follows from what we have just shown that for each n there is exactly one element $A'_n \in P^n$ with $\nu(A'_n) = 1$. It follows that there is an element $x \in \Delta$ such that

$$\{x\} = \bigcap_{n=1}^\infty A'_n$$

and $\nu(\{x\}) = 1$. Hence $m = m_x$. It follows from Corollary 4.6 that

$$m(Z(v)) = \lim_{k \rightarrow \infty} \int_{Z(v_0)} K_\beta(v, s(p_k)) \, dm_x(p) = x_v$$

for all $v \in \Gamma_V$, showing that m is concentrated on the set (5.1). \square

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